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NONTRIVIALITY OF THE STABLE HOMOTOPY ELEMENT γ_1

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0. Introduction

The p-primary stable homotopy groups of spheres, whose structure is still not well understood, are known to contain two infinite families $\{\alpha_t\}$, $\{\beta_t\}$ of nontrivial elements for each prime $p \geqslant 5$ (see [6, 9]). A parallel family of elements $\{\gamma_t\}$ can be constructed if $p \geqslant 7$ (only γ_1 is known to be constructible if $p = 5$), but whether any of them is nontrivial has been an open question. We prove:

Main Theorem. $\gamma_1 \neq 0$.

Furthermore, $\gamma_2, ..., \gamma_{p-1}$ are all nontrivial; in a forthcoming paper we will prove this and give generalizations to higher families of homotopy elements using similar methods. For some applications of $\gamma_1 \neq 0$, see [16].

When H. Toda first asked whether γ_1 was nontrivial a few years ago [12], he remarked that the question seemed very difficult. He was right; in fact, publication of these results has been delayed by a controversy between S. Oka and H. Toda and the authors (not to be confused with an earlier contradiction involving a different element named γ [10]). In October 1971, Toda answered his own conjecture: he announced at a meeting of the Japan Mathematical Society that $\gamma_1 = 0$. This result was slow in reaching the U.S.A., however, and in June 1972 the authors, unaware of Toda's announcement, completed the proof given below that $\gamma_1 \neq 0$. That August, Larry Smith, having noticed a reference to Toda's claim in a paper of Oka, alerted us to the contradiction. We found that Toda, now in collaboration with Oka, had prepared a revised version of his argument. Since then we have exchanged proofs with Oka and Toda, but no serious error has been found in either.

Recently, J.F. Adams has verified our result, using slightly different methods, in lectures at the University of Chicago. Thus we welcome the skeptical reader to inspect carefully the proof below that γ_1 is nontrivial.

The stable map γ_1 is defined as the composite in the diagram

(1)
$$\begin{tikzcd}
S^T V(2) \arrow[r, "\phi_2"] \arrow[dddr, "g_2"] & V(2) \arrow[d, "\delta_1"] \\
S^T V(1) \arrow[u, "i_1"] \arrow[ddr, "g_1"] & S^{2p^2-1} V(1) \arrow[d, "\delta_0"] \\
S^T V(0) \arrow[u, "i_0"] \arrow[dr, "g_0"] & S^{2p^2+2p-2} V(0) \arrow[d, "\delta_{-1}"] \\
S^T \arrow[u, "i_{-1}"] \arrow[r, "\gamma_1 = g_{-1}"] \arrow[d, equal] & S^{2p^2+2p-1} \\
S^T V(-1) &
\end{tikzcd}$$

Here the $V(n)$ are the spectra introduced by Smith (see [6, 11]), $T = 2(p^3 - 1)$, and the *master map* ϕ_2, which gives rise to the family $\{\gamma_t\}$ by iteration, has $V(3)$ as its mapping cone. (This holds for $p \geqslant 7$; if $p = 5$, ϕ_2 is not known to exist, but g_2 does, and the same proof works.) Thus γ_1 lies in the stem ${}_p\pi^S_{(p^2-1)q-3}$ ($q = 2(p-1)$ from now on). This stem is known to be isomorphic to Z_p, generated by $\alpha_1\beta_{p-1}$. For γ_1, then, the question is whether the above construction gives a nontrivial element; the higher γ's, lying beyond the range of known stable stems, are completely new.

The space $V(0)$ is the classical Moore space for the prime p; the map $S^q V(0) \to V(0)$ whose cone is $V(1)$ was used by Toda and Adams, but it was Larry Smith who first introduced $V(1)$ itself, and realized that the process could be generalized. He found the map $S^{pq-1} V(1) \to V(1)$ whose cone is $V(2)$, and showed that the corresponding family of stable homotopy elements is nontrivial, and (for $p > 3$) forms an infinite extension of the finite family $\beta_1, ..., \beta_{p-1}$ discovered by Toda. Smith's work was the main inspiration for the construction of the new families we will be studying in this and subsequent papers.

Our method of proof relies on the cohomology theory given by the Brown–Peterson spectrum for the prime p (see [3]). This theory, BP*, resembles ordinary cohomology (H*) more closely than K-theory, for example, yet has a much richer structure than H*. It carries just as much information at the given prime as complex cobordism, MU*, but its algebra of primary operations is much simpler. Besides, it is q-*sparse*, in the sense that the algebra of operations BP* (BP) is zero unless $* \equiv 0 \bmod q$. This sparseness allows us to contruct a third-order operation Ξ from a purely algebraic higher relation among primary operations, to compute the action of Ξ in the mapping cone $C\gamma_1$, and to express the indeterminacy in a simple form. We do this, in the stable homotopy category [2, 13], in a way that can be understood by a reader unacquainted with higher-order operations. For basic information about BP-cohomology, see [14].

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1. Constructing the tertiary operation

In ordinary cohomology theory, relations among primary operations (for example, Adem relations) give rise to secondary cohomology operations. An algebraic relation-among-relations in the Steenrod algebra need not give rise to a tertiary operation, however (see [5]). The situation for BP cohomology is better; in fact, in this section we show that any such higher relation among primary BP operations gives rise to a tertiary operation in BP cohomology. We also introduce the specific higher relation we will need to detect γ_1.

First, we recall that secondary operations (or, equivalently, stable two-stage Postnikov systems) can be constructed in *any* reasonable cohomology theory $E^*(\ )$, represented by a spectrum E. Think of a relation $\alpha_2\alpha_1 = 0$ among primary operations in E-cohomology as a sequence of stable maps $A_1 \xrightarrow{\alpha_1} A_2 \xrightarrow{\alpha_2} A_3$, with each A_i a sum of suspensions of E, and $\alpha_2\alpha_1 \sim *$ (i.e., $\alpha_2\alpha_1 = 0$ in the stable category).

Lemma 1.1. *There is a diagram*

$$\begin{tikzcd} E_2 \arrow[r, "\psi_2"] \arrow[d, "p_2"'] & S^{-1}A_3 \\ A_1 \arrow[r, "\alpha_1"] & A_2 \end{tikzcd}$$

and a map $i_2 : S^{-1}A_2 \to E_2$ *such that*

$$S^{-1}A_2 \xrightarrow{i_2} E_2 \xrightarrow{p_2} A_1 \xrightarrow{\alpha_1} A_2$$

is a stable cofibration sequence ("exact triangle" in Boardman's terminology), and $\psi_2 i_2 = S^{-1}\alpha_2$.

Proof. Let $C\alpha_1$ be the mapping cone, and let $E_2 = S^{-1}(C\alpha_1)$. Since $S^{-1}(\alpha_2\alpha_1)$ is null-homotopic, α_2 factors through E_2 as shown:

$$(2)\qquad \begin{tikzcd} S^{-1}A_1 \arrow[r, "S^{-1}\alpha_1"] & S^{-1}A_2 \arrow[r, "i_2"] \arrow[dr, "S^{-1}\alpha_2"'] & E_2 \arrow[r, "p_2"] \arrow[d, "\psi_2"] & A_1 \arrow[r, "\alpha_1"] & A_2 \\ & & S^{-1}A_3 & & \end{tikzcd}$$

This defines ψ_2 and proves the lemma.

For third-order operations we need to specialize to E = BP. Recall from [14, §3] that $[\mathrm{BP}, \mathrm{BP}]^j = 0$ unless $j \equiv 0 \bmod q$ – we say that BP is *q-sparse*. In particular, suppose

$$A_1 \xrightarrow{\alpha_1} A_2 \xrightarrow{\alpha_2} A_3 \xrightarrow{\alpha_3} A_4$$

is a nontrivial relation-among-relations in BP (that is, $\alpha_3\alpha_2 = \alpha_2\alpha_1 = 0$ and no α_i is zero). Then if we assume A_1 = BP for simplicity, we see that each A_i must be a sum of spectra of form S^{nq} BP, $n \in \mathbf{Z}$. With these data we can construct a three-stage Postnikov system.

Lemma 1.2 (Sparseness Lemma). *There is a diagram*

$$\begin{tikzcd}
S^{-2}A_3 \arrow[r, "i_3"] & E_3 \arrow[r, "\psi_3"] \arrow[d, "p_3"] & S^{-2}A_4 \\
S^{-1}A_2 \arrow[r, "i_2"] & E_2 \arrow[r, "\psi_2"] \arrow[d, "p_2"] & S^{-1}A_3 \\
 & A_1 \arrow[r, "\alpha_1"] & A_2 .
\end{tikzcd} \tag{3}$$

extending the previous diagram (2), *such that*

$$S^{-2}A_3 \xrightarrow{i_3} E_3 \xrightarrow{p_3} E_2 \xrightarrow{\psi_2} S^{-1}A_3$$

is a cofibration and $\psi_3 i_3 = S^{-2}\alpha_3$.

Proof. Since $[\mathrm{BP}, \mathrm{BP}]^i = 0$ unless $i \equiv 0 \bmod q$, it follows that $[A_1, S^{-1}A_4] = [\mathrm{BP}, S^{nq-1}\mathrm{BP}] = 0$. Because

$$S^{-1}A_2 \xrightarrow{i_2} E_2 \xrightarrow{p_2} A_1$$

is a cofibration, we have an exact sequence of groups

$$[S^{-1}A_2, S^{-1}A_4] \xleftarrow{i_2^*} [E_2, S^{-1}A_4] \xleftarrow{p_2^*} [A_1, S^{-1}A_4] ,$$

and so i_2^* is injective. Consider the composite map

$$E_2 \xrightarrow{\psi_2} S^{-1}A_3 \xrightarrow{S^{-1}\alpha_3} S^{-1}A_4 ;$$

then

$$i_2^*(S^{-1}\alpha_3\,\psi_2) = S^{-1}\alpha_3(\psi_2\, i_2) = S^{-1}(\alpha_3\,\alpha_2) = 0 .$$

Consequently, $S^{-1}\alpha_3\psi_2 = 0$, and so by Lemma 1.1 there is a secondary operation

$$\begin{tikzcd} S^{-2}A_3 \arrow[r, "i_3"] & E_3 \arrow[r, "\psi_3"] \arrow[d, "p_3"'] & S^{-2}A_4 \\ & E_2 \arrow[r, "\psi_2"] & S^{-1}A_3 \end{tikzcd}$$

such that $\psi_3 i_3 = S^{-1}(S^{-1}\alpha_3) = S^{-2}\alpha_3$. This proves Lemma 1.2.

The specific higher relation we will use is the following:

$$\begin{tikzcd} A_1 \arrow[r, "\alpha_1"] \arrow[d, equal] & A_2 \arrow[r, "\alpha_2"] \arrow[d, equal] & A_3 \arrow[r, "\alpha_3"] \arrow[d, equal] & A_4 \arrow[d, equal] \\ \mathrm{BP} \arrow[r, "\alpha_1"] & S^{pq}\mathrm{BP} \vee S^q\mathrm{BP} \arrow[r, "\alpha_2"] & S^{pq}\mathrm{BP} \vee S^{(p+2)q}\mathrm{BP} \arrow[r, "\alpha_3"] & S^{(p+1)q}\mathrm{BP}, \end{tikzcd}$$

$$\alpha_1 = \begin{pmatrix} r_p \\ r_1 \end{pmatrix}, \quad \alpha_2 = \begin{pmatrix} -p + v_1 r_1 & r_{p-1} - v_1 r_p \\ r_2 & -r_{0,1} - \tfrac{1}{2}(p+1)\, r_{p+1} + \tfrac{1}{2} v_1\, r_{1,1} \end{pmatrix},$$

$$\alpha_3 = (r_1, -2v_1).$$

For notation, see [14].

Lemma 1.3. $\alpha_2\alpha_1 = \alpha_3\alpha_2 = 0$.

The proof is postponed to §4.

We may thus define a tertiary BP operation Ξ as follows: If $u \in \mathrm{BP}^k(X) = [X, A_1]^k$ has a lifting to $\bar{u} \in [X, E_3]^k$, let $\Xi(u) = \{\psi_3 \bar{u}\} \subseteq [X, S^{-2}A_4]^k$, where $\bar{u}$ runs over all such liftings.

2. The indeterminacy

Usually, the indeterminacy of a tertiary operation is expressed in terms of a secondary operation. If we consider any nontrivial tertiary BP operation on the cone $C\gamma_1$, however, the indeterminacy turns out to be expressible in terms of primary operations. The reason, again, is sparseness.

Using the maps g_r of diagram (1) above, construct a cofiber sequence

$$S^T V(r) \xrightarrow{g_r} S^N \xrightarrow{h_r} Cg_r \xrightarrow{\rho_r} S^{T+1} V(r)$$

for $-1 \leqslant r \leqslant 2$. (Here $N = 2p^2 + 2p - 1$). There are also maps $k_r : Cg_r \to Cg_{r+1}$ $(-1 \leqslant r \leqslant 1)$ coming from the diagram

$$\begin{tikzcd} S^T V(r) \arrow[r, "g_r"] \arrow[d, "i_r"'] & S^N \arrow[r, "h_r"] \arrow[d, equal] & Cg_r \arrow[r, "\rho_r"] \arrow[d, "k_r"'] & S^{T+1} V(r) \arrow[d, "i_r"'] \\ S^T V(r+1) \arrow[r, "g_{r+1}"] & S^N \arrow[r, "h_{r+1}"] & Cg_{r+1} \arrow[r, "\rho_{r+1}"] & S^{T+1} V(r+1) \end{tikzcd} \tag{4}$$

Note that $\mathrm{BP}^*(V(2)) = S^{(p+2)q+3}\,\mathrm{BP}^*/(p, V_1, V_2)$.

Lemma 2.1.

$$\mathrm{BP}^*(C\gamma_1) = S^{T+1}\,\mathrm{BP}^* \oplus S^N\,\mathrm{BP}^*$$

with canonical generators λ_0, λ_1;

$$\mathrm{BP}^*(Cg_0) = S^{T+2}\,\mathrm{BP}^*/(p) \oplus S^N\,\mathrm{BP}^*$$

with canonical generators θ''_0, θ''_1;

$$\mathrm{BP}^*(Cg_1) = S^{T+2(p-1)+3}\,\mathrm{BP}^*/(p, v_1) \oplus S^N\,\mathrm{BP}^*$$

with canonical generators θ'_0, θ'_1;

$$\mathrm{BP}^*(Cg_2) = S^{T+2p^2+2p}\,\mathrm{BP}^*/(p, v_1, v_2, v_3) \oplus S^N(p, v_1, v_2)$$

with canonical generators θ_0, θ_1.

Proof. Use elementary exact-sequence manipulations, as in [7, 15]. Notice that θ'_1 is canonical (i.e., uniquely determined up to units of $\mathbf{Q}_p$) since $\mathrm{BP}^*/(p, v_1)$ is zero in dimension p^2q.

Observe that $T \equiv 0 \bmod q$, $N \equiv 3 \bmod q$. This is important for the following:

Corollary 2.2. *Let Ξ be any tertiary operation constructed as in Lemma 1.2. If $X = C\gamma_1$, the indeterminacy of Ξ is exactly $\alpha_3[X, S^{-2}A_3]$.*

Proof. Construct the diagram

$$\begin{tikzcd}
S^{-2}A_3 \arrow[r, "l"] & Q \arrow[r, "j"] \arrow[d, "m"] & E_3 \arrow[r, "\psi_3"] \arrow[d, "p_3"] & S^{-2}A_4 \\
 & S^{-1}A_2 \arrow[r, "i_2"] & E_2 \arrow[r, "\psi_2"] \arrow[d, "p_2"] & S^{-1}A_3 \\
 & & A_1 \arrow[r, "\alpha_1"] & A_2
\end{tikzcd}$$

where Q is $C(p_2p_3)$. By the Verdier axiom (cf. [2, p. 107]), there are maps $l: S^{-2}A_3 \to Q, m: Q \to S^{-1}A_2$ so that $jl = \pm i_3$ and the sequence

$$S^{-2}A_3 \xrightarrow{l} Q \xrightarrow{m} S^{-1}A_2 \xrightarrow{\psi_2 i_2} S^{-1}A_3$$

is a cofibration sequence (up to equivalence).

Let $u: S^{-k}X \to A_1$ represent any element of $\mathrm{BP}^*(X)$; by Lemma 2.1, $k \equiv 1$ or $3 \bmod q$. Now any two maps from X to E_3 which both lift u must differ by a map $\bar{u}: S^{-k}X \to Q$. But

$$[S^{-k}X, S^{-1}A_2] = [C\gamma_1, S^{k-1}(S^q\,\mathrm{BP} \vee S^{pq}\,\mathrm{BP})] = 0$$

by Lemma 2.1 again, since $k-1$ is congruent to 0 or 2 mod q, but $q \geqslant 8$. Hence $m\bar{u} = 0$, so $\bar{u}$ lifts to $w: X \to S^{-2}A_3$. This shows that

$$\mathrm{indet}(\Xi) = \psi_3 jl[X, S^{-2}A_3] = \psi_3\, i_3[X, S^{-2}A_3] = \alpha_3[X, S^{-2}A_3]$$

by Lemma 1.2. The converse inclusion is clear; this proves the corollary.

3. Calculating the operation Ξ in $C\gamma_1$

Our procedure now becomes similar to that used in [15] for secondary operations. We use the Peterson–Stein trick, only twice: we reduce the calculation of Ξ in $C\gamma_1$ to the calculation of a secondary operation in Cg_0 and then to the calculation of a primary operation in Cg_1.

First, notice that in view of diagram (1) above, Ck_r is equivalent to $S^{T+2(p^{r+1}-1)+2}\,V(r)$ and we have a cofibration

$$S^{T+2(p^{r+1}-1)+1}\,V(r) \xrightarrow{\zeta_r} Cg_r \xrightarrow{k_r} Cg_{r+1} \xrightarrow{\epsilon_r} S^{T+2(p^{r+1}-1)+2}\,V(r)$$

for $-1 \leqslant r \leqslant 1$.

Lemma 3.1.

$$\epsilon_1^*(\gamma) = \theta_0\,, \qquad \epsilon_0^*(\gamma') = \theta_0'\,, \qquad \epsilon_{-1}^*(\gamma'') = \theta_0''\,;$$
$$k_0^*(\theta_0') = 0\,, \qquad k_{-1}^*(\theta_0'') = 0\,;$$
$$k_0^*(\theta_1') = \theta_1''\,, \qquad k_{-1}^*(\theta_1'') = \lambda_1\,;$$
$$\zeta_1^*(\theta_0') = v_2\gamma\,, \qquad \zeta_0^*(\theta_0'') = v_1\gamma'\,, \qquad \zeta_{-1}^*(\lambda_0) = p\gamma''\,;$$
$$\zeta_1^*(\theta_1') = bv_3\gamma\,, \qquad \zeta_0^*(\theta_1'') = 0\,, \qquad \zeta_{-1}^*(\lambda_1) = 0\,,$$

where $b \not\equiv 0 \bmod p$, and the γ's are generators of the appropriate $\mathrm{BP}^*(V(r))$'s.

Proof. This is a routine calculation, using hopscotch diagrams like [15, Figs. 1 and 2]. We omit the details.

The next theorem may be viewed as a double application of either the Peterson–Stein formula or Maunder's Axiom 5 for higher operations [5] to reduce the order of the operation being studied from tertiary to secondary to primary. First, notice that $h_1^*(r_1\,\theta_1') = h_1^*(r_p\,\theta_1') = 0$ since all r_E are zero on $\mathrm{BP}^N(S^N)$. Hence $r_p(\theta_1') = a_2\,\theta_0'$, $r_1(\theta_1') = a_2'\,\theta_0'$ for some $a_2, a_2' \in \mathrm{BP}^*$, so if $w_2 = (a_2\,\gamma', a_2'\,\gamma')$, then

$$\epsilon_0^*(w_2) = (r_p\,\theta_1', r_1\,\theta_1') = \alpha_1\,\theta_1'\,.$$

Theorem 3.2 (Order Reduction Theorem). *There are elements $w_3 \in [S^{2(p^3-1)+2}, S^{-1}A_3]$, $z_3 \in \mathrm{BP}^*(C\gamma_1)$ such that*

$$\zeta_{-1}^*\,z_3 = \alpha_3\,w_3\,, \qquad \zeta_0^*\,\epsilon_{-1}^*\,w_3 = \alpha_2\,w_2$$

and

$$z_3 = \pm(-v_2^{p-2} + v_2^{p-3}\,v_1^{p+1} + (-p^2+p+1)\,v_1^{p^2-p-2})\,\lambda_0 \in \Xi(\lambda_1)\,.$$

This theorem calculates $\Xi(\lambda_1)$, up to indeterminacy. For the proof, we need:

Lemma 3.3. *For the Hazewinkel generators v_1, v_2, v_3 (see [4]),*

(a) $$\left.\begin{aligned} r_1\,v_3 &= -v_2^p \\ r_p\,v_3 &= 0 \end{aligned}\right\} \bmod p, v_1\,;$$

(b) $$\left.\begin{aligned} (r_{p-1} - v_1\,r_p)(-v_2^{p-1}) &= -v_1^{p^2-p} - v_2^{p-2}\,v_1^2 \\ (-r_{0,1} - \tfrac{1}{2}(p+1)r_{p+1} + \tfrac{1}{2}v_1\,r_{1,1})(-v_2^{p-1}) &= -v_2^{p-3}\,v_1^{p+1} \end{aligned}\right\} \bmod p\,;$$

(c) $$r_1(-v_1^{p^2-p-1} - v_2^{p-2}\,v_1) - 2v_1(-v_2^{p-3}\,v_1^p) =$$
$$= -p\,v_2^{p-2} + p\,v_2^{p-3}\,v_1^{p+1} + (-p^2+p+1)p\,v_1^{p^2-p-2}\,.$$

The proof of Lemma 3.3 is postponed to Section 4.

Reassuring remark. The correctness of the generators v_1, v_2 and v_3 has been verified separately by Liulevicius, Kozma, Araki and Adams.

Corollary 3.4. *We may take* $w_2 = (0, -v_2^{p-1}\gamma')$.

Proof of Theorem 3.2. We claim there is z_2 in $[Cg_0, S^{-1}A_3]$ such that $z_2 \in \Psi_2(\theta_1'')$, where Ψ_2 is the secondary operation constructed in Lemma 1.1 associated to the relation $\alpha_2\alpha_1 = 0$, and $\zeta_0^* z_2 = \alpha_2 w_2$. To get z_2, construct the diagram

$$\begin{tikzcd}
S^{-1}A_2 \arrow[r, "i_2"] & E_2 \arrow[r, "p_2"] & A_1 \arrow[r, "\alpha_1"] & A_2 \\
S^{T+q+1}V(0) \arrow[u, "w_2"] \arrow[r, "\zeta_0"] & Cg_0 \arrow[u, "u'"] \arrow[r, "k_0"] & Cg_1 \arrow[u, "\theta_1'"] \arrow[r, "\epsilon_0"] & S^{T+q+2}V(0) \arrow[u, "w_2"]
\end{tikzcd}$$

as follows: We have $\alpha_1(k_0^*\theta_1') = k_0^*\epsilon_0^* w_2 = 0$, so there is $u' : Cg_0 \to E_2$ lifting $k_0^*\theta_1' = \theta_1''$. It is not hard to check that u' can be chosen so that the left-hand square commutes. Since u' is a lift of θ_1'', the element $z_2 = \psi_2 u'$ lies in $\Psi_2(\theta_1'')$; also,

$$\zeta_0^* z_2 = \psi_2 u' \zeta_0 = \psi_2 i_2 w_2 = \alpha_2 w_2 .$$

So z_2 is as claimed. Now notice that

$$((-v_1^{p^2-p-1} - v_2^{p-2}v_1)\theta_0'', -v_2^{p-3}v_1^p\theta_0'') = \bar{z}$$

satisfies $\zeta_0^*\bar{z} = \alpha_2 w_2$, by Lemma 3.3(b); since ζ_0^* is injective in this grade (Lemma 3.1), $\bar{z} = z_2$.

Now pick

$$w_3 = ((-v_1^{p^2-p-1} - v_2^{p-2}v_1)\gamma'', -v_2^{p-3}v_1^p\gamma'') ;$$

then $\epsilon_{-1}^* w_3 = z_2$, and $\zeta_0^*\epsilon_{-1}^* w_3 = \alpha_2 w_2$. We now claim that there is $z_3 \in [C\gamma_1, S^{-2}A_4]$ such that $z_3 \in \Xi(\lambda_1)$, and $\zeta_{-1}^*(z_3) = \pm\alpha_3 w_3$. Construct the diagram

$$\begin{tikzcd}
S^{-2}A_3 \arrow[r, "i_3"] & E_3 \arrow[r, "p_3"] & E_2 \arrow[r, "\psi_2"] & S^{-1}A_3 \\
S^{T+1} \arrow[u, "w_3"] \arrow[r, "\zeta_{-1}"] & C\gamma_1 \arrow[u, "u''"] \arrow[r, "k_{-1}"] & Cg_0 \arrow[u, "u'"] \arrow[r, "\epsilon_{-1}"] & S^{T+2} \arrow[u, "w_3"]
\end{tikzcd}$$

The right-hand square commutes because $\psi_2 u' = z_2 = \epsilon_{-1}^* w_3$. Now $\psi_2 u' k_{-1} = w_3\epsilon_{-1}k_{-1} = 0$, so as before there is u'' (as shown) lifting $k_{-1}^* u'$, and u'' can be chosen so that the left-hand square commutes. But

$$p_2 p_3 u'' = p_2 u' k_{-1} = \theta'_1 k_0 k_{-1} = k^*_{-1} k^*_0 \theta'_1 = \lambda_1$$

by Lemma 3.1, so u'' is a lift of λ_1. Hence $z_3 = \psi_3 u'' \in \Xi(\lambda_1)$. Also,

$$\zeta^*_{-1} z_3 = \psi_3 u'' \zeta_{-1} = \psi_3 i_3 w_3 = \pm \alpha_3 w_3 .$$

Since

$$\bar{\bar{z}} = \pm(-v_2^{p-2} + v_2^{p-3} v_1^{p+1} + (-p^2 + p + 1) v_2^{p^2-p-2}) \lambda_0$$

also satisfies $\zeta^*_{-1} \bar{\bar{z}} = \pm \alpha_3 w_3$, and ζ^*_{-1} is injective in this grading, we have $z_3 = \bar{\bar{z}}$. This proves Theorem 3.2.

Lemma 3.5. *The indeterminacy of Ξ acting on λ_1 lies in $(p, v_1)\,\mathrm{BP}^*(C\gamma_1)$.*

Proof. By Corollary 2.2,

$$(\mathrm{Indet}\, \Xi) \subset \alpha_3 [C\gamma_1, S^{-2} A_3] = \mathrm{im}(r_1, -2v_1) \cap \mathrm{BP}^{(p-2)(p+1)q} .$$

Since $\mathrm{BP}^{(p^2-p-1)q}$ is generated by $v_2^{p-2} v_1, v_2^{p-3} v_1^{p+2}, \ldots$, and $r_1 v_1 = p$, $r_1 v_2 = -v_1^p \bmod p$ (see Section 4, (vi) below), the lemma follows from the Cartan formula.

Corollary 3.6. *$\Xi(\lambda_1)$ is a nonzero multiple of λ_0 modulo indeterminacy.*

Proof. $v_2^{p-2} \notin (p, v_1)$.

Now if γ_1 were null-homotopic, $C\gamma_1$ would be homotopy equivalent to $S^0 \vee S^{(p^2-1)q-2}$, and there would be maps

$$S^0 \xrightarrow{i} C\gamma_1 \xrightarrow{j} S^0$$

with $ji = \mathrm{Id}_{S^0}$. Hence $i^* j^* i^* \lambda_1 = i^* \lambda_1$, so $i^* : \mathrm{BP}^0(C\gamma_1) \to \mathrm{BP}^0(S^0)$ is iso; thus $\lambda_1 = j^* i^* \lambda_1$. We would then have

$$\Xi\lambda_1 \supset j^* i^* \Xi\lambda_1 \supset j^* i^* (v_2^{p-2} \lambda_0) = j^* 0 = 0 ,$$

a contradiction. Hence γ_1 must be essential, proving the Main Theorem.

4. Technicalities

In this section we prove Lemmas 1.3 and 3.3. We need some preliminary results.

Lemma 4.1. *With the filtration F^i of* BP*-*modules defined in* [14], $\psi\, t_k \in F^{qp^{k-1}}$. *(Here ψ is the diagonal map on the Hopf algebra* $BP_*(BP)$.)

Proof. Induction on k: For $k = 1$, we know that $\psi\, t_1 = t_1 \otimes 1 + 1 \otimes t_1$ and this lies in filtration q. Assuming the truth of the lemma for a given $k-1$, we invoke [1, Theorem 16.1(v)]:

$$\psi\, t_k = -\sum_{\substack{i+j=k\\ i>0}} m_i(\psi\, t_j)^{p^i} + \sum_{h+i+j=k} m_h(t_i)^{p^h} \otimes (t_j)^{p^{h+i}},$$

where we abbreviate m_{p^i-1} by m_i. By induction, every term $t^G \otimes t^H$ in $(\psi\, t_j)^{p^i}$ has grade $\geqslant q\,p^{j-1}(p^i) = q\,p^{k-1}$. And in the right-hand sum,

$$|t_i^{p^h} \otimes t_j^{p^{h+i}}| = 2(p^k - p^h) \geqslant 2(p^k - p^{k-1}) = q\,p^{k-1}\,.$$

This proves the lemma.

Definition. If $\psi\, t^E = \Sigma\, v^{F_i} t^{G_i}$, let the *spillover* of t^E be defined by

$$s(E) = |t^E| - \min\{s \mid \psi\, t^E \in F^s\}\,.$$

We have $s(\Delta_1) = 0$, and by Lemma 4.1,

$$s(\Delta_k) \leqslant q(p^{k-2} + p^{k-3} + \ldots + 1)$$

for $k > 1$. Clearly, $s(E_1 + E_2) = s(E_1) + s(E_2)$. The following is immediate from the definition of the multiplication in BP*(BP) in terms of the diagonal ψ:

Corollary 4.2. *If $r_E\, r_F = \Sigma\, c_K\, r_K$, then $c_K = 0$ for any K satisfying $|K| - s(K) > |E + F|$.*

This provides a way of determining when the expansion of a product $r_E\, r_F$ comes to a halt. We use it in:

Lemma 4.3. $r_a\, r_1 = (a+1) r_{a+1} - v_1\, r_{a-p+1,1}$, $0 \leqslant a \leqslant 2p-2$ *(we adopt the convention $r_{e,1} = 0$ if $e < 0$).*

Proof. Write $r_a\, r_1 = \Sigma\, c_K\, r_K$. Using the formula from [1] invoked above, we can write down $\psi\, t_1$ and $\psi\, t_2$ explicitly, and get

$$\begin{aligned} c_{a+1} &= \langle \psi\, t_1^{a+1}, r_a \otimes r_1\rangle = a+1\,,\\ c_{a-p,1} &= \langle \psi\, t_1^{a-p}\, t_2, r_a \otimes r_1\rangle = 0\,,\\ c_{a-p+1,1} &= \langle \psi\, t_1^{a-p+1}\, t_2, r_a \otimes r_1\rangle = -v_1 \quad (\text{if } a-p+1 \geqslant 0). \end{aligned}$$

Now by Corollary 4.2, $c_r = 0$ for $r > a + 1$ since $s(r\Delta_1) = 0$; and also

$$|(r,k)| - s(r,k) \geq q(r + k(p+1) - k) > q(a+1) = |r_a r_1|$$

if either $k \geq 2$ or $k = 1$ and $r > a - p + 1$ (since $a \leq 2p - 2$), so for these (r,k), $c_{r,k} = 0$. Finally, $c_E = 0$ if there is some $j > 2$ with $e_j \neq 0$ for then

$$|E| - s(E) \geq \Sigma e_k \,[(p^k - 1) - q(p^{k-2} + \ldots + 1)] = \Sigma e_k\, q\, p^{k-1}$$
$$\geq q\, e_j\, p^{j-1} \geq p^2 q > (a+1) q = |r_a r_1|.$$

This proves Lemma 4.3.

Proof of Lemma 1.3. We have to verify the three formulas

(i) $\quad w_p: \quad (-p + v_1 r_1) r_p + (r_{p-1} - v_1 r_p) r_1 = 0$

(cf. [15, Proposition 2.1]),

(ii) $\quad w_{p+2}: \quad r_2 r_p + (-r_{0,1} - \tfrac{1}{2}(p+1) r_{p+1} + \tfrac{1}{2} v_1 r_{1,1}) r_1 = 0\,.$

(iii) $\quad x_{p+1}: \quad r_1 w_p - 2v_1 w_{p+2} = 0\,.$

These all follow from the following calculations:

(iv) $\quad r_1 r_p = r_{0,1} + (p+1) r_{p+1} - v_1 r_{1,1}\,.$

(v) $\quad r_2 r_p = r_{1,1} + \binom{p+2}{2} r_{p+2} - (2 + \tfrac{1}{2}(p-1)) v_1 r_{2,1}\,.$

For example, to get (i), write

$$\begin{array}{rl}
-p\, r_p = & -p\, r_p \\
v_1 r_1 r_p = & \qquad\qquad v_1 r_{0,1} + (p+1) v_1 r_{p+1} - v_1^2 r_{1,1} \\
r_{p-1} r_1 = & p\, r_p - v_1 r_{0,1} \\
-v_1 r_p r_1 = & \qquad\qquad\qquad\quad -(p+1) v_1 r_{p+1} + v_1^2 r_{1,1} \\
\hline
 & 0
\end{array}$$

Formula (ii) follows in a similar way from Lemma 4.3, (v), and the relations $r_{0,1} r_1 = r_{1,1}$ ([14, Lemma 4.3] or direct check), and

$$\begin{aligned} r_{1,1} r_1 = (r_{0,1} r_1) r_1 = r_{0,1}(r_1 r_1) &= r_{0,1}(2r_2) && \text{(Lemma 4.3)} \\ &= 2r_{2,1} && \text{([14, 4.3])}. \end{aligned}$$

For (iii),

$$r_1 w_p = (-p r_1 + (r_1 v_1) r_1 + v_1 r_1 r_1) r_p + (r_1 r_{p-1} - (r_1 v_1) r_p - v_1 r_1 r_p) r_1 ;$$

using $r_1 v_1 = p$, $r_1 r_{p-1} = r_{p-1} r_1$ ([14, 5.10], or direct check), Lemma 4.3 and (iv), the coefficient of r_p reduces to $2v_1 r_2$, and the coefficient of r_1 becomes $-2v_1 r_{0,1} - (p+1) v_1 r_{p+1} + v_1 r_{1,1}$, proving (iii).

It remains to verify (iv) and (v). By [14, 5.10], $r_1 r_p - r_p r_1 = r_{0,1}$, so (iv) follows from Lemma 4.3. Alternatively, the formula may be checked directly using the technique of the proof of Lemma 4.3. For (v), write $r_2 r_p = \Sigma c_K r_K$. We see that $c_{p+2} = \binom{p+2}{2}$, and

$$c_{1,1} = [\text{coefficient of } t_1^2 \otimes t_1^p \text{ in } (\psi t_1)(\psi t_2)] = 1 .$$

Also

$$\psi(t_1^2 t_2) = (t_1^2 \otimes 1 + 2t_1 \otimes t_1 + 1 \otimes t_1^2) \times ((-v_1/p) \sum_{\substack{a+b=p \\ 0<a,b<p}} \binom{p}{a} t_1^a \otimes t_1^b + \text{other terms})$$

and the coefficient of $t_1^2 \otimes t_1^p$ here is $(-2 - \frac{1}{2}(p-1)) v_1$. As before, $c_r = 0$ for $r > p+2$, while for $c_{r,k}$,

$$|(r,k)| - s(r,k) \geqslant q(r + kp) > q(p+2) = |r_2 r_p|$$

if $k = 1$, $r > 2$, or if $k \geqslant 2$, so these $c_{r,k}$ vanish. All other c_E are zero for the same reason as in Lemma 4.3. This completes the proof of Lemma 1.3.

Proof of Lemma 3.3. Let $h : \mathrm{BP}^* \to H_*(\mathrm{BP})$ be the Hurewicz map; then by [4], we can choose v_3 such that

$$h(v_3) = p m_3 - h(v_1^{p^2}) m_2 - h(v_2^p) m_1 .$$

Now $r_1 m_n$ is given by [14, 3.3], and in the proof of [15, 3.7] we computed

(vi) $\qquad r_p v_2 = v_1 \bmod p , \quad r_1 v_2 = -v_1^p \bmod p , \quad r_j v_2 = 0 \bmod p$

for $2 < j < p-1$. Hence

$$\begin{aligned} h(r_1 v_3) = r_1 (h(v_3)) &= 0 - h(p^3 v_1^{p^2-1}) m_2 + h(p v_2^{p-1} v_1^p) m_1 - h(v_2^p) \bmod p^2 m_1 \\ &= h(-x v_1^{p^2-1} + v_1^{p+1} v_2^{p-1} - v_2^p) \bmod p , \end{aligned}$$

where x is defined by $h(x) = p^3 m_2$, and we have used $h v_1 = p m_1$. This proves that $r_1 v_3 = -v_2^p \bmod(p, v_1)$ since h is injective. Degree considerations show that $r_p v_3$ cannot be a power of v_2, hence is zero mod (p, v_1). This proves (a); for (b),

$$(r_{p-1} - v_1 r_p)(-v_2^{p-1}) = -((r_1 v_2)^{p-1} - v_1 (p-1)(r_p v_2) V_2^{p-2}) \bmod p$$
$$= -v_1^{p^2-p} - v_2^{p-2} v_1^2 \bmod p ,$$

by (vi) and the Cartan formula; the second part of (b) is similar, using the fact that $r_{0,1} v_2$ is divisible by p, and hence so is $r_{1,1} v_2^{p-1}$. Finally, we do (c):

$$\begin{aligned} r_1 (v_1^{p^2-p-1} - v_2^{p-2} v_1) &= (p^2 - p - 1)(r_1 v_1) v_1^{p^2-p-2} \\ &\quad - (p-2)(r_1 v_2) v_2^{p-3} v_1 - v_2^{p-2}(r_1 v_1) \\ &= -p(p^2 - p - 1) v_1^{p^2-p-2} - p v_2^{p-2} \\ &\quad + (p-2)(p+1) v_2^{p-3} v_1^{p+1} . \end{aligned}$$

The result follows.

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